Boundary Feedback Control of Complex Ginzburg-Landau Equation with A Simultaneously Space and Time Dependent Coefficient

Junji Jia

Department of Applied Mathematics, The University of Western Ontario

London, ON, N6A 5B7, Canada

Email: jjia5@uwo.ca

Abstract

Linearized complex Ginzburg-Landau equation models various physical phenomena and the stability controls of them are important. In this paper, we study the control of the LCGLE with a simultaneously space and time dependent coefficient by transforming it into a complex heat equation. It is shown that under certain conditions on the coefficient functions $a_2(\tilde{x}, \tilde{t})$, the exponential stability of the system at any rate can be achieved by boundary control based on the state feedback. The kernels are *explicitly* calculated as series of approximation and shown to be twice differentiable by using the *method of dominant*. Both the exponential stabilities of the systems with Dirichlet and Neumann boundary conditions are strictly proven.

Keywords: partial differential equations, Ginzburg-Landau equation, heat equation, boundary control, stabilization.

AMS Subject Classification. 35K05, 93D15.

1 Introduction

In this paper, we use the boundary feedback control law, which is based on the back stepping methodology, to stabilize the linearized complex Ginzburg-Landau equation (LCGLE)

$$\frac{\partial u(\tilde{x},\tilde{t})}{\partial \tilde{t}} = a_1 \frac{\partial^2 u(\tilde{x},\tilde{t})}{\partial \tilde{x}^2} + a_3(\tilde{x}) \frac{\partial u(\tilde{x},\tilde{t})}{\partial \tilde{x}} + a_2(\tilde{x},\tilde{t}) u(\tilde{x},\tilde{t}) \tag{1.1}$$

where $\tilde{x} \in (0, L),\ \tilde{t} \in (0, T),\ u : (0, L)\times(0, T) \to \mathbb{C},\ a_1 \in \mathbb{C}$ and $\Re(a_1) > 0,\ a_3 \in C^1([0, L]; \mathbb{C})$ and $a_2 \in C^1([0, L] \times [0, T]; \mathbb{C})$ is a simultaneously space and time dependent coefficient function. The Dirichlet boundary conditions of the system are

$$u(0, t) = p(t), \tag{1.2}$$

$$u(L, t) = 0 \tag{1.3}$$

where $p(t)$ is the control input. This LCGLE models various physical phenomena, such as the amplitude equation in pattern formation [2] and the reaction diffusion of two chemicals in one dimension [2, 3], in all of which the controls of the systems are important.

Without losing any generality, we set $a_3(\tilde{x}) = 0$ since this term could always be eliminated by a gauge transform

$$u \to \tilde{u} = u e^{-1/2 \int_0^{\tilde{x}} a_3(y)/a_1 dy} \tag{1.4}$$

to obtain the equation we will concentrate on hereafter

$$\frac{\partial \tilde{u}(\tilde{x},\tilde{t})}{\partial \tilde{t}} = a_1 \frac{\partial^2 \tilde{u}(\tilde{x},\tilde{t})}{\partial \tilde{x}^2} + a_2(\tilde{x},\tilde{t}) \tilde{u}(\tilde{x},\tilde{t}). \tag{1.5}$$

Equation (1.5) is equivalent to a complex linear heat equation. The feedback control of this equation where the a_1 and a_2 are real constants was first addressed by Boskovic, Krstic and Liu in [4] and the instability of the systems was also shown there. Later the plant coefficient was generalized to a space dependent case $a_2 = a_2(\tilde{x})$ by Liu [5]. More recently, Aamo, Smyshlyaev and Krstic [1, 6, 7] considered the generalization to a complex equation with a complex space or time dependent coefficient function, i.e., $a_2 = a_2(\tilde{x})$ or $a_2 = a_2(\tilde{t})$. In this paper, we extend their work by generalizing the plant coefficient to a simultaneously space and time dependent case, i.e., $a_2 = a_2(\tilde{x}, \tilde{t})$. We also considered the Neumann boundary condition problems. Besides, from the proof of lemma (2.1) we can see that the explicit result for the control kernel is given and therefore it could be immediately used in numerical simulation.

The paper is organized as follows. In section 2, we prove that for some $a_2(\tilde{x}, \tilde{t})$, the system (1.5) with the Dirichlet boundary conditions could be stabilized by the boundary feedback control. The unique existence of the kernels are shown in subsection 2.1; Using these kernels, in subsection 2.2 the original system is transformed into a new well-posed

system. In subsection 2.3 we show that the original system is exponentially stabilized. In section 3, we prove that the Neumann boundary problem can also be stabilized using the similar methodology.

2 Dirichlet Boundary Value Problem

First we break equation (1.5) into two coupled PDEs with real domains, real coefficient functions and real codomains by defining

$$\rho(x,t) = \Re(u(x,t)) = \frac{1}{2}\left(u(x,t) + \overline{u}(x,t)\right), \tag{2.1}$$

$$\iota(x,t) = \Im(u(x,t)) = \frac{1}{2i}\left(u(x,t) - \overline{u}(x,t)\right) \tag{2.2}$$

where

$$x = \frac{L-\tilde{x}}{L}, t = \frac{\tilde{t}}{T} \text{ and } u(x,t) = \tilde{u}(\tilde{x},\tilde{t}) \tag{2.3}$$

and $^{-}$ denotes the complex conjugate. Equation (1.5) then becomes

$$\rho_t = a_R\rho_{xx} + b_R(x,t)\rho - a_I\iota_{xx} - b_I(x,t)\iota, \tag{2.4}$$

$$\iota_t = a_R\iota_{xx} + b_R(x,t)\iota + a_I\rho_{xx} + b_I(x,t)\rho \tag{2.5}$$

for $(x,t) \in (0,1) \times (0,1) \equiv \Sigma$, with boundary conditions

$$\rho(0,t) = 0, \ \iota(0,t) = 0 \tag{2.6}$$

$$\rho(1,t) = p_R(t), \ \iota(1,t) = p_I(t) \tag{2.7}$$

where

$$a_R = \frac{1}{L^2}\Re(a_1), \ a_I = \frac{1}{L^2}\Im(a_1), \tag{2.8}$$

$$b_R = \Re\big(a_2(\tilde{x},\tilde{t})\big), \ b_I = \Im\big(a_2(\tilde{x},\tilde{t})\big). \tag{2.9}$$

Note that the transformation (2.3) is to normalize the x domain and the t domain. To stabilize the new equation system (2.4)-(2.9), we set the boundary control input $p(t)$ into the form

$$p_R(t) = \int_0^1 [k(1,y,t)\rho(y,t) + k_c(1,y,t)\iota(y,t)]dy \tag{2.10}$$

$$p_I(t) = \int_0^1 [k(1,y,t)\iota(y,t) - k_c(1,y,t)\rho(y,t)]dy, \tag{2.11}$$

where k and k_c are kernels that we should find.

2.1 The Unique Existence of the Kernels

First, we state a lemma about the kernels k and k_c for future use. It will be clear in lemma (2.2) why we consider this PDE system.

Lemma 2.1. *If $a_R b_R(x,t) + a_I b_I(x,t)$ is analytic in t, the partial differential equation system of k and k_c:*

$$k_{xx} = k_{yy} + \beta(x,y,t)k + \beta_c(x,y,t)k_c + p_1 k_t + p_2 k_{c,t} \tag{2.12}$$

$$k_{c,xx} = k_{c,yy} - \beta_c(x,y,t)k + \beta(x,y,t)k_c + q_1 k_{c,t} + q_2 k_t \tag{2.13}$$

for $(x,y) \in \Omega$ with boundary conditions

$$k(x,x,t) = -\frac{1}{2}\int_0^x \beta(\gamma,\gamma,t)d\gamma \tag{2.14}$$

$$k_c(x,x,t) = \frac{1}{2}\int_0^x \beta_c(\gamma,\gamma,t)d\gamma \tag{2.15}$$

$$k(x,0,t) = 0 \tag{2.16}$$

$$k_c(x,0,t) = 0 \tag{2.17}$$

where

$$p_1 = q_1 = a_R/(a_R^2 + a_I^2), \tag{2.18}$$

$$p_2 = -q_2 = -a_I/(a_R^2 + a_I^2), \tag{2.19}$$

$$\beta(x,y,t) = [a_R(b_R(y,t) - f_R(x,t)) + a_I(b_I(y,t) - f_I(x,t))]/(a_R^2 + a_I^2), \tag{2.20}$$

$$\beta_c(x,y,t) = [a_R(b_I(y,t) - f_I(x,t)) - a_I(b_R(y,t) - f_R(x,t))]/(a_R^2 + a_I^2), \tag{2.21}$$

in which $f_R(x,t)$, $f_I(y,t)$ defined in theorem (2.1) are analytic in t and differentiable in y, has a unique solution satisfying

$$|k(x,y,t)| \leq Me^{M(x^2-y^2)} \tag{2.22}$$

$$|k_c(x,y,t)| \leq Me^{M(x^2-y^2)}\ , \tag{2.23}$$

where M is a positive constant, for any time interval $(0,t_0)(t_0 < 1)$ we are concerning.

Proof. Using the substitutions

$$x = \xi + \eta,\ y = \xi - \eta, \tag{2.24}$$

$$G(\xi,\eta,t) = k(x,y,t),\ G_c(\xi,\eta,t) = k_c(x,y,t), \tag{2.25}$$

$$\delta(\xi,\eta,t) = \beta(x,y,t),\ \delta_c(\xi,\eta,t) = \beta_c(x,y,t), \tag{2.26}$$

we can have

$$\begin{array}{ll} k_{xx} = 1/4(G_{\xi\xi} + 2G_{\xi\eta} + G_{\eta\eta}) & k_{c,xx} = 1/4(G_{c,\xi\xi} + 2G_{c,\xi\eta} + G_{c,\eta\eta}) \\ k_{yy} = 1/4(G_{\xi\xi} - 2G_{\xi\eta} + G_{\eta\eta}) & k_{c,yy} = 1/4(G_{c,\xi\xi} - 2G_{c,\xi\eta} + G_{c,\eta\eta}) \\ k_t = G_t & k_{c,t} = G_{c,t}. \end{array} \tag{2.27}$$

Thus the system (2.12)-(2.17) is transformed into

$$G_{\xi\eta}(\xi,\eta,t) = \left(\delta(\xi,\eta,t) + p_1\frac{\partial}{\partial t}\right)G(\xi,\eta,t) + \left(\delta_c(\xi,\eta,t) + p_2\frac{\partial}{\partial t}\right)G_c(\xi,\eta,t) \tag{2.28}$$

$$G_{c,\xi\eta}(\xi,\eta,t) = \left(\delta(\xi,\eta,t) + q_1\frac{\partial}{\partial t}\right)G_c(\xi,\eta,t) + \left(-\delta_c(\xi,\eta,t) + q_2\frac{\partial}{\partial t}\right)G(\xi,\eta,t) \tag{2.29}$$

with boundary conditions

$$G(\xi,0,t) = -\frac{1}{2}\int_0^{\xi}\delta(\tau,0,t)\delta\tau, \tag{2.30}$$

$$G_c(\xi,0,t) = \frac{1}{2}\int_0^{\xi}\delta_c(\tau,0,t)\delta\tau, \tag{2.31}$$

$$G(\xi,\xi,t) = 0, \tag{2.32}$$

$$G_c(\xi,\xi,t) = 0. \tag{2.33}$$

By direct integrating of (2.28) and (2.29) with respect to ξ and η and using (2.30)-(2.31), we obtain the equivalent integral equations

$$\begin{aligned} G(\xi,\eta,t) = -\frac{1}{2}\int_{\eta}^{\xi}\delta(\tau,0,t)d\tau + \int_{\eta}^{\xi}\int_0^{\eta}\left(\delta(\tau,s,t) + p_1\frac{\partial}{\partial t}\right)G(\tau,s,t)dsd\tau \\ + \int_{\eta}^{\xi}\int_0^{\eta}\left(\delta_c(\tau,s,t) + p_2\frac{\partial}{\partial t}\right)G_c(\tau,s,t)dsd\tau \end{aligned} \tag{2.34}$$

$$\begin{aligned} G_c(\xi,\eta,t) = \frac{1}{2}\int_{\eta}^{\xi}\delta(\tau,0,t)d\tau + \int_{\eta}^{\xi}\int_0^{\eta}\left(\delta(\tau,s,t) + q_1\frac{\partial}{\partial t}\right)G_c(\tau,s,t)dsd\tau \\ + \int_{\eta}^{\xi}\int_0^{\eta}\left(-\delta_c(\tau,s,t) + q_2\frac{\partial}{\partial t}\right)G(\tau,s,t)dsd\tau \end{aligned} \tag{2.35}$$

To proceed, we set

$$G_0(\xi,\eta,t) = -\frac{1}{2}\int_{\eta}^{\xi}\delta(\tau,0,t)\delta\tau,\ G_{c,0}(\xi,\eta,t) = \frac{1}{2}\int_{\eta}^{\xi}\delta(\tau,0,t)\delta\tau \tag{2.36}$$

$$\begin{aligned} G_{n+1}(\xi,\eta,t) = \int_{\eta}^{\xi}\int_0^{\eta}\left(\delta(\tau,s,t) + p_1\frac{\partial}{\partial t}\right)G_n(\tau,s,t)dsd\tau \\ + \int_{\eta}^{\xi}\int_0^{\eta}\left(\delta_c(\tau,s,t) + p_2\frac{\partial}{\partial t}\right)G_{c,n}(\tau,s,t)dsd\tau \end{aligned} \tag{2.37}$$

$$G_{c,n+1}(\xi,\eta,t) = \int_\eta^\xi\int_0^\eta \left(\delta(\tau,s,t) + q_1\frac{\partial}{\partial t}\right) G_{c,n}(\tau,s,t)dsd\tau$$
$$+ \int_\eta^\xi\int_0^\eta \left(-\delta_c(\tau,s,t) + q_2\frac{\partial}{\partial t}\right) G_n(\tau,s,t)dsd\tau. \tag{2.38}$$

We want to show that both the series

$$G = \sum_{n=1}^{\infty} G_n(\xi,\eta,t) \text{ and } G_c = \sum_{n=1}^{\infty} G_{c,n}(\xi,\eta,t) \tag{2.39}$$

are uniquely convergent and thus by the method of successive approximation they are the solutions of system (2.34)-(2.35).

This could be done by the virtue of Colton's *method of dominant* [8, 9]. This method works as follows. If we are given two series

$$S_1 = \sum_{n=1}^{\infty} a_{1n}t^n,\ S_2 = \sum_{n=1}^{\infty} a_{2n}t^n,\ t\in(0,1), \tag{2.40}$$

where $a_{2n}\geq 0$, then we say S_2 *dominates* S_1 if $|a_{1n}|\leq a_{2n},\ n=1,2,3,\cdots$, and write $S_1 \ll S_2$. It can be easily checked that

$$\text{if } S_1 \ll S_2, \qquad \text{then } |S_1| \leq S_2, \tag{2.41}$$
$$\text{and } \frac{\partial S_1}{\partial t} \ll \frac{\partial S_2}{\partial t},\ S_1 \ll S_2(1-t)^{-1}; \tag{2.42}$$
$$\text{if } S_1 \ll S_2, S_2 \ll S_3, \qquad \text{then } S_1 \ll S_3; \tag{2.43}$$
$$\text{if } S_1 \ll S_2, S_3 \ll S_4, \qquad \text{then } S_1 + S_2 \ll S_3 + S_4. \tag{2.44}$$

Moreover, the property of *dominancy* can also be kept if the integrals of the two series are not with respect to t but other variables, that is

$$\text{if } S_1 \ll S_2,\ \text{then } \int_a^b S_1dx \ll \int_a^b S_2dx. \tag{2.45}$$

On the other hand, if a function f is analytic in $t\in(0,1)$, then there exist a positive constant C such that $f \ll C(1-t)^{-1}$.

Using this method in our problem, it can be shown that equations (2.12) and (2.13) have unique twice continuously differentiable solutions if δ and δ_c are analytic in t. In fact, since $a_Rb_R(x,t)+a_Ib_I(x,t)$ is analytic in t and $f_I(x,t)$ and $f_R(x,t)$ can also be chosen to be analytic in t, from equation (2.20), (2.21) and (2.26), we know that δ and δ_c are analytic in t. Thus we can let $C,\ C_c$ be two positive constants larger than 1 such that

$$\delta(\xi,\eta,t) \ll C(1-t)^{-1},\ \delta_c(\xi,\eta,t) \ll C_c(1-t)^{-1}, \tag{2.46}$$

or further

$$\delta(\xi,\eta,t) \ll N(1-t)^{-1},\ \ \delta_c(\xi,\eta,t) \ll N(1-t)^{-1},\ \ N = \max\{C, C_c\}. \tag{2.47}$$

From equation of (2.36) and noticing (2.45), we have

$$G_0 \ \ll \ N(1-t)^{-1} \tag{2.48}$$

$$G_{c,0} \ \ll \ N(1-t)^{-1}. \tag{2.49}$$

By induction, suppose

$$G_n \ \ll \ \frac{4^n(\xi\eta)^n N^{n+1}}{n!}(1-t)^{-(n+1)} \tag{2.50}$$

$$G_{c,n} \ \ll \ \frac{4^n(\xi\eta)^n N^{n+1}}{n!}(1-t)^{-(n+1)}, \tag{2.51}$$

we will have from (2.37)

$$\begin{aligned} G_{n+1} =& \int_\eta^\xi\int_0^\eta \left(\delta + p_1\frac{\partial}{\partial t}\right) G_n ds d\tau + \int_\eta^\xi\int_0^\eta \left(\delta_c + p_2\frac{\partial}{\partial t}\right) G_{c,n} ds d\tau \\ \ll& \int_\eta^\xi\int_0^\eta \left(\frac{N}{1-t} + p_1\frac{\partial}{\partial t}\right) \frac{4^n(s\tau)^n N^{n+1}}{n!}(1-t)^{-(n+1)} ds d\tau \\ &+ \int_\eta^\xi\int_0^\eta \left(\frac{N}{1-t} + p_2\frac{\partial}{\partial t}\right) \frac{4^n(s\tau)^n N^{n+1}}{n!}(1-t)^{-(n+1)} ds d\tau \\ \ll& \frac{4^n(\xi\eta)^{n+1}N^{n+1}}{((n+1)!)^2}(1-t)^{-(n+2)}(2N + np_1 + np_2) \\ \ll& \frac{4^{n+1}(\xi\eta)^{n+1}N^{n+2}}{(n+1)!}(1-t)^{-(n+2)}, \end{aligned} \tag{2.52}$$

where all the properties (2.41)-(2.45) have been used. Similarly, we also have

$$G_{c,n+1} \ll \frac{4^{n+1}(\xi\eta)^{n+1}N^{n+2}}{(n+1)!}(1-t)^{-(n+2)} \tag{2.53}$$

and hence by (2.41)

$$|G_{n+1}| \leq \frac{4^{n+1}(\xi\eta)^{n+1}N^{n+2}}{(n+1)!}(1-t)^{-(n+2)}\ , |G_{c,n+1}| \leq \frac{4^{n+1}(\xi\eta)^{n+1}N^{n+2}}{(n+1)!}(1-t)^{-(n+2)}. \tag{2.54}$$

It is clear that the two series convergent absolutely and uniformly and are solution of (2.34)-(2.35). G and G_c are C^2 since $\delta(\xi,\eta,t)$ and $\delta_c(\xi,\eta,t)$ are C^1. Since

$$4^n(\xi\eta)^n N^n(1-t)^{-n} = \left((x^2-y^2)\frac{N}{1-t}\right)^n, \tag{2.55}$$

(2.22) and (2.23) follow directly by assigning $N/(1-t_0) = M$. □

Remark 2.1. *(1). The proof of Lemma (2.1) provides a numerical computation scheme of successive approximation to compute the kernels. This makes the feedback laws very useful in real problems. (2). We require $t \in (0, t_0)$, $t_0 < 1$ in (2.22) and (2.23) since from (2.55) one can see that k and k_c are not bounded when $t \to 1$. This requirement is tolerable since in practice, we only require the system to be stable in a time interval $\tilde{t} \in (0, T_0)$ where $T_0 = t_0 T$ from rescaling equation (2.3) and the T can be choose as large as we want.*

2.2 Conversion and Inverse Convertibility of the Original System

We want to show that the original system (2.4)-(2.11) can be converted to a new system by integral transform with k and k_c as kernels and this conversion is invertible.

Lemma 2.2. *Let $k(x, y, t)$ and $k_c(x, y, t)$ be the solution of (2.12)-(2.13) and define a pair of linear bounded operator K and K_c by*

$$\tilde{\rho}(x,t) = (K\rho)(x,t) = \rho(x,t) - \int_0^x [k(x,y,t)\rho(y,t) + k_c(x,y,t)\iota(y,t)]dy \tag{2.56}$$

$$\tilde{\iota}(x,t) = (K_c\iota)(x,t) = \iota(x,t) - \int_0^x [-k_c(x,y,t)\rho(y,t) + k(x,y,t)\iota(y,t)]dy, \tag{2.57}$$

Then,
1.K and K_c convert system (2.4)-(2.11) to system

$$\tilde{\rho}_t = a_R\tilde{\rho}_{xx} + f_R(x,t)\tilde{\rho} - a_I\tilde{\iota}_{xx} - f_I(x,t)\tilde{\iota}, \tag{2.58}$$

$$\tilde{\iota}_t = a_I\tilde{\rho}_{xx} + f_I(x,t)\tilde{\rho} + a_R\tilde{\iota}_{xx} + f_R(x,t)\tilde{\iota}, \tag{2.59}$$

for $x \in (0,1)$, with boundary conditions

$$\tilde{\rho}(0,t) = 0,\ \tilde{\iota}(0,t) = 0,\ \tilde{\rho}(1,t) = 0,\ \tilde{\iota}(1,t) = 0. \tag{2.60}$$

2. Both K and K_c have linear bounded inverses.

Proof. Proof of part 1.: To prove *1*, we compute as follows. Differentiating (2.56) and using (2.4)-(2.5), we have

$$\begin{aligned}\tilde{\rho}_t(x,t) =& a_R\rho_{xx} + b_R(x,t)\rho - a_I\iota_{xx} - b_I(x,t)\iota \\ &- \int_0^x [k(x,y,t)(a_R\rho_{xx} + b_R(y,t)\rho - a_I\iota_{xx} - b_I(y,t)\iota) + k_t\rho \\ &\quad k_c(x,y,t)(a_I\rho_{xx} + b_I(y,t)\rho + a_R\iota_{xx} + b_R(y,t)\iota) + k_{c,t}\iota]dy.\end{aligned}$$

Using (2.56)-(2.57), integrating the integral by parts twice and using (2.6), (2.16)-(2.17) and (2.60), we will get

$$
\begin{aligned}
\tilde{\rho}_t(x,t) =& a_R \left(\tilde{\rho}_{xx}(x,t) + \frac{\partial^2}{\partial x^2} \int_0^x [k(x,y,t)\rho(y,t) + k_c(x,y,t)\iota(y,t)]dy \right) \\
&+ b_R(x,t) \left(\tilde{\rho}(x,t) + \int_0^x [k(x,y,t)\rho(y,t) + k_c(x,y,t)\iota(y,t)]dy \right) \\
&- a_I \left(\tilde{\iota}_{xx}(x,t) + \frac{\partial^2}{\partial x^2} \int_0^x [-k_c(x,y,t)\rho(y,t) + k(x,y,t)\iota(y,t)]dy \right) \\
&- b_I(x,t) \left(\tilde{\iota}(x,t) + \int_0^x [-k_c(x,y,t)\rho(y,t) + k(x,y,t)\iota(y,t)]dy \right)
\end{aligned}
$$

$$
\begin{aligned}
&- k(x,x,t)a_R\rho_x(x,t) + k(x,x,t)a_I\iota_x(x,t) - k_c(x,x,t)a_I\rho_x(x,t) - k_c(x,x,t)a_R\iota_x(x,t) \\
&+ k_y(x,x,t)a_R\rho(x,t) - k_y(x,x,t)a_I\iota(x,t) + k_{c,y}(x,x,t)a_I\rho(x,t) + k_{c,y}(x,x,t)a_R\iota(x,t) \\
&- \int_0^x [k_{yy}(x,y,t)(a_R\rho(y,t) - a_I\iota(y,t)) + k(x,y,t)(b_R(y,t)\rho(y,t) - b_I(y,t)\iota(y,t)) \\
&\quad + k_{c,yy}(x,y,t)(a_I\rho(y,t) + a_R\iota(y,t)) + k_c(x,y,t)(b_I(y,t)\rho(y,t) + b_R(y,t)\iota(y,t)) \\
&\quad + k_t(x,y,t)\rho(y,t) + k_{c,t}(x,y,t)\iota(y,t)]dy.
\end{aligned} \tag{2.61}
$$

Using the identity

$$
\begin{aligned}
\frac{\partial^2}{\partial x^2} \int_0^x \lambda(x,y,t)\chi(y,t)dy =& \int_0^x \lambda_{xx}(x,y,t)\chi(y,t)dy + \lambda_x(x,x,t)\chi(x,t) \\
&+ \chi(x,t)\frac{\partial \lambda(x,x,t)}{\partial x} + \lambda(x,x,t)\chi_x(x,t)
\end{aligned} \tag{2.62}
$$

and the equation (2.58)-(2.59) again, we obtain

$$
\begin{aligned}
\tilde{\rho}_t(x,t) =& a_R\tilde{\rho}_{xx}(x,t) - a_I\tilde{\iota}_{xx}(x,t) + b_R(x,t)\tilde{\rho}(x,t) - b_I(x,t)\tilde{\iota}(x,t) \\
&+ 2\left(a_R\frac{\partial k(x,x,t)}{\partial x} + a_I\frac{\partial k_c(x,x,t)}{\partial x} \right)\tilde{\rho}(x,t) + \int_0^x R(x,y,t)\rho(y,t)dy \\
&+ 2\left(a_R\frac{\partial k_c(x,x,t)}{\partial x} - a_I\frac{\partial k(x,x,t)}{\partial x} \right)\tilde{\iota}(x,t) + \int_0^x I(x,y,t)\iota(y,t)dy
\end{aligned} \tag{2.63}
$$

where

$$
\begin{aligned}
R(x,y,t) =& a_R(k_{xx}(x,y,t) - k_{yy}(x,y,t)) + a_I(k_{c,xx}(x,y,t) - k_{c,yy}(x,y,t)) - k_t(x,y,t) \\
&+ 2\left(a_R\frac{\partial k(x,x,t)}{\partial x} + a_I\frac{\partial k_c(x,x,t)}{\partial x} + b_R(x,t) - b_R(y,t) \right) k(x,y,t) \\
&+ 2\left(-a_R\frac{\partial k_c(x,x,t)}{\partial x} + a_I\frac{\partial k(x,x,t)}{\partial x} + b_I(x,t) - b_I(y,t) \right) k_c(x,y,t),
\end{aligned} \tag{2.64}
$$

$$
\begin{aligned}
I(x,y,t) =& a_R(k_{c,xx}(x,y,t) - k_{c,yy}(x,y,t)) - a_I(k_{xx}(x,y,t) - k_{yy}(x,y,t)) - k_{c,t}(x,y,t) \\
&+ 2\left(a_R \frac{\partial k(x,x,t)}{\partial x} + a_I \frac{\partial k_c(x,x,t)}{\partial x} + b_R(x,t) - b_R(y,t)\right) k_c(x,y,t) \\
&+ 2\left(a_R \frac{\partial k_c(x,x,t)}{\partial x} - a_I \frac{\partial k(x,x,t)}{\partial x} - b_I(x,t) + b_I(y,t)\right) k(x,y,t).
\end{aligned} \tag{2.65}
$$

Substituting equation (2.12)-(2.13) and (2.14)-(2.15) into (2.64)-(2.65), we can get

$$
\begin{aligned}
&R(x,y,t) = [(a_R p_1 + a_I q_2) - 1]k_t(x,y,t) + (a_R p_2 + a_I q_1)k_{c,t}(x,y,t)) \\
&+ (a_R\beta(x,y,t) - a_I\beta_c(x,y,t) - (a_R\beta(x,x,t) - a_I\beta_c(x,x,t)) + b_R(x,t) - b_R(y,t))k(x,y,t) \\
&+ (a_R\beta_c(x,y,t) + a_I\beta_c(x,y,t) - (a_R\beta_c(x,x,t) + a_I\beta(x,x,t)) + b_I(x,t) - b_I(y,t))k_c(x,y,t),
\end{aligned} \tag{2.66}
$$

$$
\begin{aligned}
&I(x,y,t) = [(a_R q_1 - a_I p_2) - 1]k_{c,t}(x,y,t) + (a_R q_2 + a_I p_1)k_t(x,y,t)) \\
&+ (-a_R\beta_c(x,y,t) - a_I\beta(x,y,t) + (a_R\beta_c(x,x,t) + a_I\beta(x,x,t)) - b_I(x,t) + b_I(y,t))k(x,y,t) \\
&+ (a_R\beta(x,y,t) - a_I\beta_c(x,y,t) - (a_R\beta(x,x,t) - a_I\beta_c(x,x,t)) + b_R(x,t) - b_R(y,t))k_c(x,y,t).
\end{aligned} \tag{2.67}
$$

From (2.20)-(2.21), we have

$$
a_R\beta(x,y,t) - a_I\beta_c(x,y,t) = b_R(y,t) - f_R(x,t) \tag{2.68}
$$

$$
a_R\beta_c(x,y,t) + a_I\beta(x,y,t) = b_I(y,t) - f_I(x,t) \tag{2.69}
$$

and with the help of (2.18)-(2.19), it is easy to check that

$$
R(x,y,t) = I(x,y,t) \equiv 0. \tag{2.70}
$$

Thus equation (2.63) becomes the following after using (2.14), (2.15) again,

$$
\begin{aligned}
\tilde{\rho}(x,t) =& a_R\tilde{\rho}_{xx}(x,t) - a_I\tilde{\imath}_{xx}(x,t) \\
&+ (-a_R\beta(x,x,t) + a_I\beta_c(x,x,t) + b_R(x,t))\tilde{\rho}(x,t) \\
&+ (a_R\beta_c(x,x,t) + a_I\beta(x,x,t) - b_I(x,t))\tilde{\imath}(x,t).
\end{aligned} \tag{2.71}
$$

Substituting (2.68) and (2.69) into above, we can obtain (2.58). Equation (2.59) follows similarly by differentiating (2.57) and applying the same procedure. The boundary conditions (2.60) follow from setting $x = 0$ and $x = 1$ in (2.56) and (2.57) and using (2.6)-(2.7) and (2.10)-(2.11). Thus we proved part *1*.

Proof of part 2.: To prove *2*, the invertibility of the transform, consider the following transform

$$\rho(x,t) = (K^{-1}\tilde{\rho})(x,t) = \tilde{\rho}(x,t) - \int_0^x [l(x,y,t)\tilde{\rho}(y,t) + l_c(x,y,t)\tilde{\iota}(y,t)]dy \quad (2.72)$$

$$\iota(x,t) = (K_c^{-1}\tilde{\iota})(x,t) = \tilde{\iota}(x,t) - \int_0^x [-l_c(x,y,t)\tilde{\rho}(y,t) + l(x,y,t)\tilde{\iota}(y,t)]dy,. \quad (2.73)$$

They are actually the inverse transform if there exist unique solutions for $l(x,y,t)$ and $l_c(x,y,t)$ that satisfying

$$l_{xx} = k_{yy} - \beta(y,x,t)k - \beta_c(y,x,t)k_c - p_1 k_t - p_2 k_{c,t} \quad (2.74)$$

$$l_{c,xx} = k_{c,yy} + \beta_c(y,x,t)k - \beta(y,x,t)k_c - q_1 k_{c,t} - q_2 k_t \quad (2.75)$$

with boundary conditions

$$l(x,x,t) = \frac{1}{2}\int_0^x \beta(\gamma,\gamma,t)d\gamma \quad (2.76)$$

$$l_c(x,x,t) = -\frac{1}{2}\int_0^x \beta_c(\gamma,\gamma,t)d\gamma \quad (2.77)$$

$$l(x,0,t) = l_c(x,0,t) = 0, \quad (2.78)$$

and

$$|l(x,y,t)| \leq Me^{M(x^2-y^2)}, \quad (2.79)$$

$$|l_c(x,y,t)| \leq Me^{M(x^2-y^2)}\ , \quad (2.80)$$

where M is a positive constant, for any time interval $(0,t_0)(t_0 < 1)$ we are concerning.

The proof that the transform (2.72) and (2.73) will lead to system (2.74)-(2.78) is similar to the proof of part *1* of this lemma. It is also clear that the transforms are linear. The unique existence of the solution to system (2.74)-(2.80) is similar to the proof of lemma (2.1) and thus the transforms are also bounded. Thus this proved that there exist the bounded linear inverse transforms as shown in (2.72) and (2.73). □

2.3 Stability of the Controlled System

Our purpose of all doing this is to control the original system by the boundary inputs. Its stability is established in the following theorem.

Theorem 2.1. *There exist feedback kernels* $k(1,\cdot,t),\ k_c(1,\cdot,t) \in C^{2,\infty}(0,1)\times(0,t_0), t_0 < 1$, *such that for arbitrary initial data* $\rho^0(x),\ \iota^0(x) \in H^1(0,1)$, *system (2.4)-(2.7) with (2.11)-(2.10), where* $b_R(x,t)$ *and* $b_I(x,t)$ *are analytic in* t, *has a unique solution* $\rho,\ \iota \in C^{2,\infty}(0,1)\times(0,t_0)$ *that is exponentially stable in the* $L_2(0,1)$ *and* $H_1(0,1)$ *norms.*

Proof. First we notice that problem (2.4)-(2.7) with (2.11)-(2.10) is well posed since, by lemma (2.2) they can be transformed into the problem (2.58)-(2.60) via the isomorphism defined by (2.56) and (2.57) and problem (2.58)-(2.60) is well posed (see, e.g. [10]).

Since through part *2* of lemma (2.2), we showed that there exist linear bounded invertible transforms, it is sufficient to show that $(\tilde{\rho}, \tilde{\iota})$ are exponentially stable. The proof of this could be done by the same definitions and evaluations of the energy and potential as in corollary 7 of Ref. [1]. Especially the $f_I(x,t), f_R(x,t)$ used in lemma (2.1) are also defined there. $\square$

3 Neumann Boundary Value Problem

To stabilize the Neumann boundary value problem

$$\begin{aligned} \rho_t &= a_R\rho_{xx} + b_R(x,t)\rho - a_I\iota_{xx} - b_I(x,t)\iota, && (3.1)\\ \iota_t &= a_R\iota_{xx} + b_R(x,t)\iota + a_I\rho_{xx} + b_I(x,t)\rho && (3.2)\end{aligned}$$

for $x \in (0,1), t \in (0,1)$, with boundary conditions

$$\begin{aligned} \rho_x(0,t) &= 0,\ \iota_x(0,t) = 0 && (3.3)\\ \rho(1,t) &= p_R(t),\ \iota(1,t) = p_I(t)\ , && (3.4)\end{aligned}$$

the transform as (2.56)-(2.57) with kernel $k(x,y,t)$ and $k_c(x,y,t)$ in the following system

$$\begin{aligned} k_{xx} &= k_{yy} + \beta(x,y,t)k + \beta_c(x,y,t)k_c + p_1k_t + p_2k_{c,t} && (3.5)\\ k_{c,xx} &= k_{c,yy} - \beta_c(x,y,t)k + \beta(x,y,t)k_c + q_1k_{c,t} + q_2k_t && (3.6)\end{aligned}$$

for $(x,y) \in \Omega$ with boundary conditions

$$\begin{aligned} k(x,x,t) &= -\frac{1}{2}\int_0^x \beta(\gamma,\gamma,t)d\gamma && (3.7)\\ k_c(x,x,t) &= \frac{1}{2}\int_0^x \beta_c(\gamma,\gamma,t)d\gamma && (3.8)\\ k_x(x,0,t) &= k_{c,x}(x,0,t) = 0 && (3.9)\\ k(0,0,t) &= k_c(0,0,t) = 0 && (3.10)\end{aligned}$$

where $\beta, \beta_c, p_1, p_2, q_1, q_2, f_I$ and f_R are the same as in lemma (2.1), will lead to the same system as (2.58)-(2.60). If we can show that the above kernels uniquely exist, all the other lemmas and theorem in section 2 can be applied in this case. The main result are established as follows, in which the unique existence of the new kernels are explicitly shown.

Lemma 3.1. *If $a_R b_R(x,t)+a_I b_I(x,t)$ is analytic in t, the partial differential equation system (3.5)-(3.10) has a unique solution satisfying*

$$|k(x,y,t)| \leq Me^{M(x^2-y^2)} \tag{3.11}$$
$$|k_c(x,y,t)| \leq Me^{M(x^2-y^2)}, \tag{3.12}$$

where M is a positive constant for $t \in (0,t_0)$ we are concerning.

Proof. The proof of this lemma is similar to that of lemma (2.1) except we should find a $G(\xi,\xi,t)$ as (2.32) in lemma (2.1) since this must be used in order to obtain $G(\xi,\eta,t)$ as (2.34). Using the same substitutions as in (2.24)-(2.26), system (3.7)-(3.8) can be transformed into

$$G_{\xi\eta} = \left(\delta(\xi,\eta,t)+p_1\frac{\partial}{\partial t}\right)G(\xi,\eta,t)+\left(\delta_c(\xi,\eta,t)+p_2\frac{\partial}{\partial t}\right)G_c(\xi,\eta,t) \tag{3.13}$$
$$G_{c,\xi\eta} = \left(\delta(\xi,\eta,t)+q_1\frac{\partial}{\partial t}\right)G_c(\xi,\eta,t)+\left(-\delta_c(\xi,\eta,t)+q_2\frac{\partial}{\partial t}\right)G(\xi,\eta,t) \tag{3.14}$$

with boundary conditions

$$G(\xi,0,t) = -\frac{1}{2}\int_0^\xi \delta(\tau,0,t)d\tau, \tag{3.15}$$
$$G_c(\xi,0,t) = \frac{1}{2}\int_0^\xi \delta_c(\tau,0,t)d\tau, \tag{3.16}$$
$$G_\xi(\xi,\xi,t) = G_\eta(\xi,\xi,t), \tag{3.17}$$
$$G_{c,\xi}(\xi,\xi,t) = G_{c,\eta}(\xi,\xi,t) \tag{3.18}$$
$$G(0,0,t) = G_c(0,0,t)=0. \tag{3.19}$$

Differentiating equation (3.15) with respect to ξ gives

$$G_\xi(\xi,0,t) = -\frac{1}{2}\delta(\xi,0,t). \tag{3.20}$$

Integrating equation (3.13) with respect to η from 0 to ξ and using (3.20) gives

$$\begin{aligned}
&G_\xi(\xi,\xi,t)\\
=&G_\xi(\xi,0,t)+\int_0^\xi\left[\left(\delta(\xi,s,t)+p_1\frac{\partial}{\partial t}\right)G(\xi,s,t)+\left(\delta_c(\xi,s,t)+p_2\frac{\partial}{\partial t}\right)G_c(\xi,s,t)\right]ds\\
=&-\frac{1}{2}\delta(\xi,0,t)+\int_0^\xi\left[\left(\delta(\xi,s,t)+p_1\frac{\partial}{\partial t}\right)G(\xi,s,t)+\left(\delta_c(\xi,s,t)+p_2\frac{\partial}{\partial t}\right)G_c(\xi,s,t)\right]ds.
\end{aligned} \tag{3.21}$$

Thus using (3.17), we have

$$
\begin{aligned}
&\frac{\partial G(\xi,\xi,t)}{\partial \xi}\\
=&G_\xi(\xi,\xi,t)+G_\eta(\xi,\xi,t)=2G_\xi(\xi,\xi,t)\\
=&-\delta(\xi,0,t)+2\int_0^\xi\left[\left(\delta(\xi,s,t)+p_1\frac{\partial}{\partial t}\right)G(\xi,s,t)+\left(\delta_c(\xi,s,t)+p_2\frac{\partial}{\partial t}\right)G_c(\xi,s,t)\right]ds.
\end{aligned}
\tag{3.22}
$$

Integrating the above equation from 0 to ξ and using (3.19) gives the function we wanted

$$
\begin{aligned}
&G(\xi,\xi,t)\\
=&-\int_0^\xi \delta(\tau,0,t)d\tau\\
&+2\int_0^\xi\int_0^\tau\left[\left(\delta(\tau,s,t)+p_1\frac{\partial}{\partial t}\right)G(\tau,s,t)+\left(\delta_c(\tau,s,t)+p_2\frac{\partial}{\partial t}\right)G_c(\tau,s,t)\right]dsd\tau.
\end{aligned}
\tag{3.23}
$$

Using (3.23) and integrating twice (3.13) first with respect to η from 0 to η and second with respect to ξ from η to ξ, we can have the following integral equation

$$
\begin{aligned}
&G(\xi,\eta,t)\\
=&-\int_0^\eta \delta(\tau,0,t)d\tau\\
&+2\int_0^\eta\int_0^\tau\left[\left(\delta(\tau,s,t)+p_1\frac{\partial}{\partial t}\right)G(\tau,s,t)+\left(\delta_c(\tau,s,t)+p_2\frac{\partial}{\partial t}\right)G_c(\tau,s,t)\right]dsd\tau\\
&+\int_\eta^\xi\int_0^\eta\left[\left(\delta(\tau,s,t)+p_1\frac{\partial}{\partial t}\right)G(\tau,s,t)+\left(\delta_c(\tau,s,t)+p_2\frac{\partial}{\partial t}\right)G_c(\tau,s,t)\right]dsd\tau.
\end{aligned}
\tag{3.24}
$$

Similarly, for $G_c(\xi,\eta,t)$ we will have

$$
\begin{aligned}
&G_c(\xi,\eta,t)\\
=&\int_0^\eta \delta(\tau,0,t)d\tau\\
&+2\int_0^\eta\int_0^\tau\left[\left(\delta(\tau,s,t)+q_1\frac{\partial}{\partial t}\right)G_c(\tau,s,t)+\left(-\delta_c(\tau,s,t)+q_2\frac{\partial}{\partial t}\right)G(\tau,s,t)\right]dsd\tau\\
&+\int_\eta^\xi\int_0^\eta\left[\left(\delta(\tau,s,t)+q_1\frac{\partial}{\partial t}\right)G_c(\tau,s,t)+\left(-\delta_c(\tau,s,t)+q_2\frac{\partial}{\partial t}\right)G(\tau,s,t)\right]dsd\tau.
\end{aligned}
\tag{3.25}
$$

Similarly as in lemma (2.1), the above integral equations can also be shown to have uniquely and uniformly convergent solution satisfying (3.11) and (3.12). □

Lemma 3.2. *Let $k(x,y,t)$ and $k_c(x,y,t)$ be the solution of (3.5)-(3.6) and define a pair of linear bounded operator K and K_c by*

$$
\tilde{\rho}(x,t) = \rho(x,t)-\int_0^x[k(x,y,t)\rho(y,t)+k_c(x,y,t)\iota(y,t)]dy \tag{3.26}
$$

$$
\tilde{\iota}(x,t) = \iota(x,t)-\int_0^x[-k_c(x,y,t)\rho(y,t)+k(x,y,t)\iota(y,t)]dy, \tag{3.27}
$$

Then,
1.k and k_c convert system (3.1)-(3.4) to system

$$\begin{aligned}\tilde{\rho}_t &= a_R\tilde{\rho}_{xx} + f_R(x,t)\tilde{\rho} - a_I\tilde{\iota}_{xx} - f_I(x,t)\tilde{\iota}, && (3.28)\\ \tilde{\iota}_t &= a_I\tilde{\rho}_{xx} + f_I(x,t)\tilde{\rho} + a_R\tilde{\iota}_{xx} + f_R(x,t)\tilde{\iota}, && (3.29)\end{aligned}$$

for $x \in (0,1)$, with boundary conditions

$$\tilde{\rho}(0,t) = 0, \tilde{\iota}(0,t) = 0, \tilde{\rho}(1,t) = 0, \tilde{\iota}(1,t) = 0. \tag{3.30}$$

2. Both K and K_c have linear bounded inverses.

Proof. The proof is similar to the proof of lemma (2.2). □

Theorem 3.1. *There exist feedback kernel $k(1,\cdot,t)$, $k_c(1,\cdot,t) \in C^{2,\infty}(0,1) \times (0,t_0), t_0 < 1$, such that for arbitrary initial data $\rho^0(x)$, $\iota^0(x) \in H^1(0,1)$, system (3.1)-(3.4) with (2.11)-(2.10) has a unique solution ρ, $\iota \in C^{2,\infty}(0,1) \times (0,t_0)$ that is exponentially stable in the $L^2(0,1)$ and $H^1(0,1)$ norms.*

Proof. The proof is similar to the proof of theorem (2.1). □

4 Remarks

A much more challenging but also more important task is to stabilize the nonlinear heat equation with a $\tilde{u}(\tilde{x},\tilde{t})$ dependent coefficient

$$\frac{\partial \tilde{u}(\tilde{x},\tilde{t})}{\partial \tilde{t}} = a_1 \frac{\partial^2 \tilde{u}(\tilde{x},\tilde{t})}{\partial \tilde{x}^2} + a_2(\tilde{x},\tilde{t},\tilde{u}(\tilde{x},\tilde{t}))\tilde{u}(\tilde{x},\tilde{t}), \tag{4.1}$$

which models virous diffusion processes with a self-dependent source for the diffused matter by using the boundary feedback control method. It could be interesting to study the applicability of the *method of dominant* in this case.

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